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ABSTRACT

This chapter builds on the idea that classrooms that promote understanding can and do exist. Elaborating upon this idea the question is raised, "In what ways can many more classrooms be developed so that all students have the opportunity to learn with understanding?" The construction of classrooms that promote understanding is dependent upon thoughtful, knowledgeable teachers who have participated in professional development programs that enable the development of their own understanding of mathematics, students' thinking about mathematics, and the interdependence of the two. (SOE)

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Chapter 10

Creating Classrooms That Promote Understanding

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The classrooms described in earlier chapters, which were developed by teachers and researchers working as teams over a long period of time, provide existence proofs that classrooms that promote understanding can and do exist. These classrooms were dramatically different from traditional ones, and students who studied in them did come to understand mathematics (see suggested reading list provided with each chapter for more information about student learning). Even with the evidence that such classrooms can exist, a critical question remains: In what ways can many more classrooms be developed so that all students have the opportunity to learn with understanding? The purpose of this chapter is to point the way toward such development.

A REPRISE

The Mathematics To Be Understood

Learning or understanding trivial, nonuseful mathematics makes no sense at all. Our concern throughout this book has been with presenting examples not only of students' learning with understanding but of students learning mathematics worthy of understanding. The definition of mathematics worthy of understanding is more complex than and different from what has usually been taught during the 20th century. Although the do-

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mains of concern appear to be the usual ones (whole numbers, fractions, geometry, etc.), the various instructional examples provided in this book make clear that these domains have been substantially redefined as being not only mathematical rules and procedures but also sets of intra- and interrelated processes and concepts.

Included in the definition of school mathematics is also the ways students think about it, understand it, and manifest their understanding. For example, in Cognitively Guided Instruction, one of the programs described in Carpenter et al. (chapter 4, this volume), addition and subtraction of whole numbers is not portrayed as two related operations but as an organized taxonomy of structurally different word problems that adults would probably solve using traditional addition or subtraction procedures. Although children solve the problems in their own invented ways, their solutions have recognizable similarities related to the taxonomy of word problems. Thus, students' thinking is clearly seen in the definition of the domain of addition–subtraction for the primary-school curriculum. In classrooms that promote mathematical understanding, students' thinking is an integral part of the mathematics curriculum.

The tasks that carry mathematics to the students portray mathematics as integrated and creatable. For instruction, the mathematics is presented in problem contexts that require mathematization and the creation of solution procedures by the students. As students engage in solving the various problems, they concurrently learn basic skills, concepts, and ways to engage in mathematical activity.

Students' Mental Activities

In the descriptions of classrooms where students are learning with understanding, instruction is not portrayed as the presentation of clear, precise explanations of procedures to be practiced by students. Instead, understanding is constructed by each student as she or he engages in the various mental activities we have found to be critical: actively constructing relationships between and among mathematical ideas by reflecting on problem solutions, extending knowledge by relating the new solutions to what has been known previously, and articulating thinking about the mathematics they have explored. By engaging in these activities, each learner assumes ownership of the mathematical knowledge that she or he has constructed.

Although each mental activity is described separately, in reality they are not isolated but integrated. *Constructing relationships* without thinking about or *reflecting* on those relationships is difficult. *Articulation* facilitates reflection because describing (or articulating) how a problem is solved requires thinking about (or reflecting on) what has been done. Because these mental activities are personal, the *understanding that develops becomes the learner's own.*

So we come full circle: Understanding is constructed, reflected on, and articulated by the learner and the knowledge that results is his or her own.

The School Curriculum

Children enter school with a rich understanding of whole numbers and an informal knowledge of space. The out-of-school environment continues to provide some school-age learners with experiences that are mathematical, but the knowledge gained from out-of-school experiences is limited and provides only a preliminary foundation for learning. Because of this limitation, the rich understanding of mathematics necessary to function in our modern society is almost always developed in formal schooling. Thus, it falls on school mathematics programs (aided and abetted by related courses) to provide the experiences necessary for learning mathematics. Although all students are dependent on school-based experiences for learning mathematics, traditionally underachieving populations, because they typically do not have a mathematically enriched environment outside of school, are even more dependent on classrooms to provide a curriculum that enables them to learn mathematics with understanding.

The Standards (NCTM, 1989) have provided as complete a description of a curriculum that would promote understanding as is available today, and it is unnecessary to provide another description here. However, there are a few critical elements of the curriculum that we wish to emphasize.

Because the goal of mathematics education should be the development of understanding by all students, the majority of the curriculum should be composed of tasks that provide students with problem *situations*. Two reasons support this claim. The first is that mathematics that is worth learning is most closely represented in problem-solving tasks. The second is that students are more apt to engage in the mental activities required to develop understanding when they are confronted with mathematics embedded in problem situations.

Tasks that students do in order to learn mathematics should be relevant to them and engage each student in a situation in which she or he can identify a problem and figure out her or his own solutions. The mathematics involved in doing the tasks should be related to other mathematical ideas in the curriculum, lead to model building, and require inquiry and justification.

Consider some of the tasks in which learners were engaged in the previous chapters. In Carpenter et al. (chapter 4, this volume), children constructed their own algorithms to solve word problems. Children in the Sowder chapter worked with division of fractions. Lehrer et al. (chapter 5, this volume) described children investigating geometric ideas, specifically the concept of triangle, which led to the study of quilt patterns. Lajoie

(chapter 7, this volume) reported on students' use of a computer database to construct a scatterplot of temperatures. Students in one of the examples presented by Kaput (chapter 8, this volume) studied functions by comparing the changes in plants' heights and the height's rate of change. These tasks varied dramatically in their complexity, the amount of time needed to complete them, the mathematical domain in which they were situated, the tools selected to be used, and the representations used (symbols or words). But in each instance the tasks were embedded in problem situations, and the students were engaged in complex mental activities that would lead to understanding.

Normative Practices of Classrooms

Normative practices are the modus operandi of all classrooms, and some norms are particularly critical for classrooms that promote understanding. One is the establishment of understanding as the major characteristic of learning and doing mathematics. When this norm is firmly established, students, in part because they are aware of its importance to the teacher, develop a predisposition to gain understanding, and understanding becomes important to them. Another classroom norm that students come to expect is articulation of problem solutions or thinking. When thinking is articulated regularly, patterns of thinking develop that are iterative. Thinking cannot be articulated unless students reflect on the problem and the strategies they use to solve it; articulation, in turn, increases reflection, which leads to understanding. Students learn to anticipate sharing their strategies, and discourse about mathematics becomes commonplace. Particularly rich examples of discourse and sharing were described earlier in the Carpenter et al., Kaput, and Lajoie (chapters 4, 7, and 8, this volume).

TEACHERS AND DEVELOPING UNDERSTANDING

Although teachers and researchers collaborated on producing the classrooms described previously, it is clear that teachers hold the major responsibility for developing similar classrooms across the nation. They are the ones who decide what is done on a day-by-day basis in their classrooms. Even with the plethora of constraints that exist, the final decision making about instructional activities is done by teachers. In this section, we take a closer look at the responsibilities of the teacher for developing classrooms that promote understanding. We then examine one professional development program that supports teachers in developing their own understandings of instruction and of their students' thinking. An examination of teachers' responsibilities helps to define the teacher's role.

Teachers' Responsibilities

The Establishment of Classroom Norms That Encourage Learning With Understanding. Because groups of students vary from year to year, as do teachers' skills in classroom management, it is impossible to write a prescription that tells a teacher how to establish classroom norms for promoting understanding. However, just as students' learning is continually being assessed, teachers also need to continually assess the norms in their classroom to ensure that the norms being manifested enable students to engage in problem-solving, reflect, and articulate their thinking.

The Selection of Tasks That Lead to Understanding. Tasks carry the mathematics and provide opportunities for problem solving, reflection, and articulation. They have to be embedded in mathematics worthy of learning and be organized into a coherent set that leads to rich understanding of a domain. Not only must teachers select tasks appropriate for their students, they also must provide appropriate tools that enable students to solve problems and to develop understanding. These can range from very simple counters to paper and pencil to complex computer programs.

The Purposeful Incorporation of Equity Concerns. Ensuring that *all* students learn with understanding involves attending to individuals while at the same time being aware of group differences that may lead to differential participation in classroom activities and learning. Group differences may have to do with the relevance of the context in which each task is embedded (as illustrated by Secada and Berman, chapter 3, this volume), appropriate social–cultural norms (such as challenging or not challenging another's thinking in discourse), or the degree of competitiveness (e.g., some girls may be hesitant to compete, whereas some boys may be hesitant to work cooperatively). There are no simple guidelines for making instructional decisions about groups of learners. But the mental activities elaborated on in this book can be encouraged in a large variety of settings, and sensitive teachers can adapt tasks and norms so that all learners can engage in those mental activities.

The Assessment of Each Student's Understanding in Both the Short and Long Term in Order to Ensure That Understanding Continues to Develop. Assessment is done for at least two reasons: so that the teacher can learn whether the tasks and the classroom structure are effective in developing students' understanding, and so that the student can gauge her or his own understanding.

Assessment can be done formally through testing and informally as students articulate and share their thinking. Classrooms that promote understanding rely heavily on informal assessment. Assessing understanding

is never particularly easy and usually requires combining knowledge from a number of sources: oral reporting or discussion, logs students might keep, daily homework, and–or formal tests. However, when students are aware that each assessment stresses understanding, not memory, they become more aware of understanding and exert greater efforts to understand.

Overall, the role of a teacher in a classroom that promotes understanding combines being very active sometimes with being almost passive at other times. The teacher must be active in establishing a classroom with norms that encourage a climate of understanding, in selecting tasks that incorporate important mathematics appropriate for each student and that will enable their understanding to grow, in assessing each student's growth, and in ensuring that all students are enabled to learn mathematics with understanding. However, sometimes a teacher must be passive and let students struggle either alone or with each other in the problem-solving process. It takes restraint on a teacher's part when the student appears to be heading in the wrong direction, but the best role of the teacher may be to just listen to what the student is doing and to ask questions that enable the student to find her or his own pathway to successful problem solving. On the other hand, sometimes the wise role for a teacher is to be active and lead the student to a solution. It takes a sensitive teacher to know when to be passive and when to actively help. She or he just needs to always remember that the goal at all times is to enable the student to increase her or his understanding.

Teachers' Knowledge

One of the most critical influences on the instructional decisions made by teachers is the knowledge that they have about mathematics and their students' mathematical thinking. These two kinds of knowledge are intertwined and their impact on students permeates the classrooms that have been described in this book. In each case, teachers had mathematical knowledge and understood their students' thinking.

Consider the case of Carmen Curtis (Lehrer et al., chapter 5, this volume) leading second-grade students to derive a mathematical definition of triangle. As she led the children in a lengthy discussion, her knowledge of mathematics and knowledge of children's thinking were manifest in the complex interactions that took place. It is clear from very early in the vignette that she knew that the mathematical definition of triangle included more than a triangle being a polygon with 3 sides and 3 angles (or a set of 3 nonlinear points connected by line segments). She also anticipated that most of her students' prior knowledge about triangles was probably limited to triangles that had 3 equal sides and 3 equal angles (an equilateral triangle).

By sensitive questioning, Ms. Curtis led the students to move in the direction of identifying relevant properties such as straight sides (rather than curved ones) and ignoring irrelevant ones such as orientation. She led them to reason about classes of triangles defined by sets of properties. She recognized the immature ideas of some children and worked with the ideas as building blocks as more mature ideas about triangles were developed. It should also be noted that this was not an isolated task but was embedded in an ongoing set of lessons. (Consider Romberg and Kaput's discussion (chapter 1, this volume) about the selection of tasks to make a coherent curriculum.) During the remainder of the school year, she continued to provide triangle-based tasks such as studying nets, building quilts, and constructing solids with triangles as components.

In the case of Ms. A, reported by Sowder and Philipp (chapter 6, this volume), the integration of the teacher's knowledge of mathematics and students' thinking is well illustrated. Her knowledge of her students can be seen in the thoughtful questions she asked, which elicited answers that revealed student thinking. Her knowledge of mathematics was particularly salient as she carefully led her students to an understanding of the referent unit of the fraction involved in fraction problems (see Sowder and Philipp, chapter 6, this volume, for a thorough discussion).

Professional Development of Teachers

The teachers of the classrooms described in earlier chapters gained their integrated knowledge of mathematics and children's thinking in a variety of ways that are beyond the scope of this book. In many cases, teachers participated in a professional development program consisting of seminars in which mathematics and research-based information about students' thinking within specific mathematical domains were discussed. Concurrently with attending seminars, teachers selected instructional tasks that required problem solving and had their students invent and report their solutions for the problems. The teachers discussed those solutions in the seminars that led to teachers' increased knowledge about their students' thinking.

Consider one such professional development program directed by Sowder and Philipp (chapter 6, this volume). In this program, middle-school teachers met regularly in seminars over a 2-year period. The purpose of the seminars was to provide the teachers with opportunities for open, reflective discussions of mathematical ideas with peers and mathematics-education scholars. During the seminars, there were many discussions of mathematical and pedagogical considerations, and vignettes from two of these discussions are presented here.

Vignette 1. Midway through the first year, a teacher brought to the seminar group a question about a student's answer to a division problem, and the group discussed the meanings of quotients and remainders. In the dialogue that follows, a teacher (T1) had just told the group about a student who had said that $37 \div 5$ was 7 remainder $\frac{2}{5}$. In the portion of the discussion included here, the teacher struggled to understand, with the researcher-leader's scaffolding, the difference between $7\frac{2}{5}$ and 7 remainder $\frac{2}{5}$:

T1: I was expecting 7 remainder 2, but I knew 7 remainder $\frac{2}{5}$ was OK too, but I didn't know for sure why. It was kind of hard to explain to him [the student] that he had the right answer.

Leader: Are those two the same: 7 remainder 2 and 7 remainder $\frac{2}{5}$? For $35\frac{1}{2} \div 5$ is 7 remainder $\frac{1}{2}$ correct?

T1: Yes.

T2: But that's not the remainder.

T3: But the thought process is right.

Leader: Let's talk about the difference between 7 remainder 2, 7 remainder $\frac{2}{5}$, and $7\frac{2}{5}$ and what they all mean. Could they all be correct or not?

T1: $37 \div 5 = 7\frac{2}{5}$. Wouldn't that be the same as 7 remainder $\frac{2}{5}$?

Leader: Is it? $7\frac{2}{5}$ is a number. It is a quantity. It is the answer. If you say 7 remainder $\frac{2}{5}$, then does it mean the same thing? If I parceled out 5 groups of 7 and had 2 left, do I divide the 2 up and share them? If I do, the answer is $7\frac{2}{5}$. Or do I just put the two aside and not consider them a part of the sharing situation? Then it's 7 remainder 2.

T1: But the real answer, the correct answer would be 7 and $\frac{2}{5}$ of 5 because that would be 2. This is the first time a student came up to me and said I did this. I had to think about it first. It was tempting to say, "You're right. Okay, I'll let you go."

T3: I have to really think about this and what it means, because the book gives the answer *Remainder* 2, and then when we get into a word problem where we need the remainder to be expressed in another way, it is hard for students to understand. The book doesn't give enough examples for you to be able to determine what's the most appropriate way to write the remainder and what does it mean in the context. Thinking about that now, I'm going to approach it a lot differently.

The discussion continued for another few minutes, looking at other examples, before the time was up for the day.

This vignette highlights some of the five forms of mental activity discussed by Carpenter and Lehrer (chapter 2, this volume). The mental activity of reflecting about mathematical experience calls for conscious examination of the role that activity and thought play in learning. The teachers were reflecting on what they were learning and that their reflections went beyond trying to make sense of this task as learners; they were also reflecting on how their students would make sense of the task.

In describing the mental activity of making knowledge one's own in chapter 2, Carpenter and Lehrer recognized the need for learners to take responsibility for their own learning by striving to understand. These teachers were personally invested in this task because the task was generated from a question asked of one of the teachers. The teachers were not simply learning by listening to the leaders but instead were taking responsibility for their own learning by questioning the leaders and each other. The mental activity of making knowledge one's own also means that learners must adopt a stance that knowledge is evolving. This notion does not apply only to children; the researcher–leaders approached all of the interactions with the teachers from this stance. This point of view led the leaders to recognize some months later, while reviewing the seminar transcripts, that the issues raised in Vignette 1 had not been fully resolved by the teachers and that the topic needed to be revisited.

Vignette 2. The following year, the same example shown in Vignette 1 was presented, but expanded as follows:

For 37 ÷ 5, make up a word problem
where the answer is 7;
where the answer is 8;
where the answer is 2;
where the answer is 7;
where the answer is 7.4;
where the answer is 7 remainder 2.

The first part of the discussion excerpted here focused on making a word problem for 37 ÷ 5 where the answer is $7\frac{2}{5}$:

T4: Something with a pie that you're dividing up. How much pie would we eat all together? If you can serve 5 people from one pie and there are 37 people in the class, how much pie will we need?

T5: There are 37 inches of ribbon. We need five bows. How long could each bow be? It's still not a good problem because you

wouldn't have a bow that's $7\frac{2}{5}$ inches long. It's not long enough to make a bow.

T3: If the idea was to get the fullest skirt, and we have 37 yards of material. What's the most material each person could use for a skirt?

Leader: Why is it that the earlier problem situations involving kids aren't working for $7\frac{2}{5}$?

T3: You can't cut kids in half.

T5: You can't have $\frac{2}{5}$ of a person.

Later in the seminar, the group discussed problems where the answer would be stated as 7 remainder 2, the most commonly given answer to $37 \div 5$ but the most difficult to interpret within a context. In this portion of the discussion, teachers returned to the problem from the earlier seminar:

Leader: So when is the answer 7 remainder 2?

T5: It would be like the bus problem, 37 kids and 5 buses—7 remainder 2. How many are left over?

Leader: But then wouldn't the answer be just 2?

T4: But when you're talking about division, that's the most common answer you'll get—7 remainder 2. You hear that more often, 'cause that's how we taught kids to say it.

T5: Unless you're using a calculator. Maybe it's not as common now that kids have calculators.

Leader: What's the difference between $7\frac{2}{5}$ and 7 remainder 2? I'll put it in context. We have 37 cookies. We're going to share them among five girls. You could say the answer is 7 remainder 2 or $7\frac{2}{5}$. What would be the difference?

T3: In the $\frac{2}{5}$, you're going to take what's left over and split it up among all the girls. When you have remainder 2, that means those two you're just going to leave for the teacher.

Leader: So in the case of $7\frac{2}{5}$, tell me what the 7 means.

T5: In that case, it means that 5 is your thing that you're trying to distribute the 37 into. That's the number of pieces. And you can do that seven times.

Leader: Are those two [in the remainder] being distributed or not?

T5: They're not being distributed in the first case [7 remainder 2]. Whereas in the other one, they are.

Leader: So in the second case, this 7 and the $\frac{2}{5}$, it sounds like they—

T4: Everybody gets $7\frac{2}{5}$.

Leader: [T1], do you remember last year when we discussed this? We talked about whether the answer to 37 ÷ 5 could be written as 7 remainder $\frac{2}{5}$.

T1: But $\frac{2}{5}$ is part of the quotient. It's not the remainder. If we say 7 remainder $\frac{2}{5}$, then each person got 7 cookies, and there was $\frac{2}{5}$ of a cookie left. But there are 2 cookies left. Why? What did I say before?

At this point, we could conclude that the teachers understand and can distinguish clearly between $7\frac{2}{5}$, 7 remainder 2, and 7 remainder $\frac{2}{5}$, as answers to 37 ÷ 5. However, when discussing the mental activity of reflecting about mathematical experience, Carpenter and Lehrer pointed out that not all relationships are equally productive, and learning with understanding involves developing relationships that reflect important principles.

What were the important principles in this vignette? To the leaders, there were three. First, the leaders wanted the teachers to grapple with different interpretations of division. Second, the leaders wanted the teachers to consider how the unit was affected during multiplication and division. Third, the leaders wanted the teachers to recognize the effect that providing a context had on the kind of reasoning they, and subsequently their students, were likely to find useful as they worked through these tasks. Carpenter and Lehrer noted in chapter 2 that articulation requires reflection in order to lift out the critical ideas of an activity so that they become objects of thought. The leaders' three principles were critical ideas, and at this point the leaders wanted to support the teachers by making the three principles explicit.

The leaders, therefore, turned the conversation to the interpretations of division. An earlier seminar had focused on meanings of all four operations. For division, two interpretations had been discussed: a partitive, or sharing, interpretation (e.g., 12 candies being shared by 3 children, with the question being the number of candies each child received); and a measurement, or repeated subtraction, interpretation (e.g., 12 candies shared so that each child received 3 candies, with the question being the number of children receiving candies, i.e., How many times can 3 be subtracted from 12?). The new element in the dialogue that follows was the meaning of the remainder in these two types of situations:

Leader: When you think about the interpretations of division, measurement and partitive, which is this? Thirty-seven cookies shared among five?

T4: That's partitive.

T6: Yeah. The other would be 37 cookies, give 5 to each student.

T5: How many students can get 5 cookies?

Leader: Now there really is a remainder 2, isn't there? Can the answer be $7\frac{2}{5}$ now?

T6: No, because that would mean you have $7\frac{2}{5}$ people.

Leader: Does 7 remainder 2 make some sense now?

T3: Yes. You know, I really appreciate the insight you have given me on measurement and partitive division. It's something I really didn't think about before. But it's so important

Leader: Suppose we go back to the bows. Let's say they are going to be 5 inches long each, what do the two numbers mean—the remainder 2 and the 7 in that problem?

T5: The number of bows you make is the 7. And the remainder is how much is excess?

The final segment of this seminar transcript dealt with the referent units for the quotient and the remainder. The group had discussed referent units in other contexts. For example, in $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$, the referent for $\frac{3}{4}$ is $\frac{1}{2}$ (we are finding $\frac{3}{4}$ of $\frac{1}{2}$), whereas the referent for the $\frac{1}{2}$ and for the $\frac{3}{8}$ is one whole. The teachers had come to realize that many of the errors made by their students were made because they did not understand referent units:

Leader: So there's an interesting question about the referents here. The remainder and the quotient refer to two different things completely. The 7 refers to the number of bows, the 2 to the number of inches left.

T7: And $\frac{2}{5}$ could mean something there. You can write $\frac{2}{5}$ or point 4.

T8: But if you were talking about ribbon, you would say, "And I have 2 inches that I can't use." I don't think you would say, "I can make $7\frac{2}{5}$ bows."

Leader: The point being made is worth repeating. If we have 37 cookies and we're sharing them among five people, we have 7 cookies for each person and 2 cookies left. In both cases the units are cookies. In this case, this is how much each of us get [7], and this [2] is how much is left over. But if we talk about how many 5-inch bows can be made with 37 inches, you end up with seven bows and 2 inches of material left. They don't even refer to the same units of measure anymore. Let's go back to cookies and cookies, giving each person five cookies. It seems that when you're doing a partitive division, then you come out with the same unit of measure, and that's why they can be combined into $7\frac{2}{5}$ or 7.4. But when you're using the measurement interpretation—

T7: But it can still be combined. You can still have $7\frac{2}{5}$ bows.

T5: But people don't think like that. When you're using that measurement model, the unit itself is what's most important, so that anything that's leftover is not one of those. Because 5 is what makes it, so what's leftover—you don't even talk about it. I wouldn't say "$\frac{2}{5}$ of a bow" because that's meaningless.

T9: If this were flour, you could combine parts of cups together, but you can't combine pieces of fabric and have one piece of fabric.

Leader: But you see, this discussion that we're having now can only be had if you're talking about a situation. It can't be had if you're just dividing numbers. And most of the time the kids in school just have numbers—they don't have situations to talk about. So there's really nothing to talk about at all.

T5: That's true of any computation that you do out of context.

Leader: But it's especially troubling with division because of the remainders. Kids don't have any idea what to do with those.

Discussion. Teachers come to understand the teaching–learning process by engaging in the same mental activities that are essential for students to learn with understanding. Consider how these seminars encouraged the teachers to engage in the various types of mental activity discussed by Carpenter and Lehrer in chapter 2. They note that although ideas take on meaning from the ways that they relate to other ideas, learning must go beyond relating ideas to existing knowledge by creating or constructing rich, integrated knowledge structures (relationships). Vignettes 1 and 2 portray not only the struggle to construct a pedagogically sound understanding of division with remainders but also show how the teachers began to connect ideas of quotients and remainders with different interpretations of division and to consider the role of referent units in division setting. Carpenter and Lehrer also suggested that applications provide an important context for developing meaning. In Vignette 2, no contexts had been used and interpretation of the remainder appeared difficult, but setting the fraction problems in real-world contexts helped clarify many of the issues raised. Third, the seminars provided an atmosphere in which the teachers could articulate their thinking and reflect on their own learning. The outcome of these mental activities for them was that they took ownership of the mathematics they were responsible for teaching.

Fourth, the process by which learners come to articulate their thinking about an idea (another mental activity) requires that learners have time to consider and reconsider the idea in relation to what they already know about the relationships constructed (through the first mental activity dis-

cussed). The normative practice of the seminar leaders of staying with a problem until its many ramifications had been explored and of probing through provocative questions and examples did not pass without notice by the teachers. The success of the discussions and the increased understanding the teachers felt for the mathematics led them to examine their own questioning strategies and, in effect, gave them permission to stay with a problem for longer periods of time in their own classrooms, even returning to it when it seemed profitable to do so. The teachers claimed that they could now manage and even enjoy these long classroom discussions for two reasons: (a) They had begun to listen to their students and to focus on what students understood rather than on what they themselves wanted to teach, and (b) they were no longer afraid of the responses they would get when they asked hard questions.

Both seminars began with seemingly simple questions surrounding the task of dividing 37 by 5. The ensuing discussions show, however, that the surrounding issues are actually quite complex. The mathematics of focus for these seminars was the mathematics of the classroom. This focus allowed the teachers to explore, in depth, the mathematics that they taught and to think about the difficulties their own students had with these concepts. We contend that focusing on either students' thinking or on mathematical concepts alone will not be enough to empower teachers to make meaningful instructional changes. Teachers need to experience a meaningful interplay between the two, an interplay that is more difficult to attain as the mathematical concepts become more complex and more interrelated.

IN CONCLUSION

A major assumption has permeated all of the work reported in this book: The goal of mathematics education is that all students learn worthy mathematics with understanding that increases throughout schooling. This book provides a set of existence proofs that classrooms that promote understanding of worthy mathematics can be developed. They are exciting places for observers, more exciting for teachers, and perhaps the most exciting for those fortunate students who are learning mathematics in them.

It is clear that teachers were the main developers of the classrooms. However, the authors as researchers participated with the teachers to construct real classrooms that coped with the variety of problems faced in schools. As we worked with teachers to understand and apply the research-based knowledge we had, we learned as much as if not more than the teachers about the teaching–learning process that enables understanding to occur. And we became convinced that the construction of classrooms that promote understanding is dependent on thoughtful, knowledgeable teach-

ers who have participated in professional development programs that enable the development of their own understanding of mathematics, students' thinking about mathematics, and the interdependence of the two. When this happens, classrooms that promote understanding will proliferate.

And the learning will not stop there. The research base that forms the foundation of teaching and learning with understanding will grow so that *all* students will learn with understanding. Sufficient classrooms that promote understanding could produce a generation of citizens who are not satisfied with procedural knowledge but seek instead to understand how mathematics can help them solve the complex issues they will be facing.

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